k-Step Sum and m-Step Gap Fibonacci Sequence

Definition: For the integers $k = 1, 2, \ldots$, $m = 0, 1, \ldots$, we define the *k-step sum and m-step gap Fibonacci sequence* $\left\{ f_n^{(k,m)}, \quad n = 1, 2, ... \right\}$, whose n-th term is given by the recurrence relation

$$f_n = f_{n-m-1} + f_{n-m-2} + \cdots + f_{n-m-(k-1)} + f_{n-m-k} = \sum_{j=1}^{k} f_{n-m-j}, \quad \text{for every } n \geq k + m + 1$$

with

$$f_1 = ... = f_{k+m} = 1.$$

This recurrence formula generates the n-th term of the sequence as the sum of k successive previous terms starting the sum at the m-th previous term.

Comments :

- For $k = 1$ and $m = 0, 1, \ldots$, all the terms of the 1-*step sum and m-step gap Fibonacci sequence* $\left\{ f_n^{(1,m)}, \quad n = 1, 2, ... \right\}$ are equal to one.
- The *k-step sum* and *m-step gap Fibonacci sequence* $\left\{ f_n^{(k,m)}, \quad n = 1, 2, ... \right\}$ gives known sequences for various values of the steps k, m:

  ✓ for $k = 2$, $m = 0$, $\left\{ f_n^{(2,0)}, \quad n = 1, 2, ... \right\}$ gives the well-known Fibonacci sequence, 1,1,2,3,5,8,13,… [3, A000045].

  ✓ for $k = 3$, $m = 0$, $\left\{ f_n^{(3,0)}, \quad n = 1, 2, ... \right\}$ is the tribonacci sequence, 1,1,1,3,5,9,17,31,… [3, A000213].

  ✓ for $k = 4$, $m = 0$, the 4-*step sum and* 0-*step gap Fibonacci sequence* $\left\{ f_n^{(4,0)}, \quad n = 1, 2, ... \right\}$ is the tetranacci sequence, 1,1,1,1,4,7,13,25,… [3, A000288].

  ✓ for $k = 2$, $m = 1$, the 2-*step sum and* 1-*step gap Fibonacci sequence* $\left\{ f_n^{(2,1)}, \quad n = 1, 2, ... \right\}$ gives the Padovan sequence, 1,1,1,2,2,3,4,5,7,9,… [3, A000931].

- The *k-step sum* and *m-step gap Fibonacci sequence* $\left\{ f_n^{(k,m)}, \quad n = 1, 2, \ldots \right\}$ is associated with the $(k+m)\times(k+m)$ matrix:

$$F_{k,m} = \begin{bmatrix} F_1 & F_2 \\ F_3 & F_4 \end{bmatrix} = \begin{bmatrix} 0 & \ldots & 0 & 1 & \ldots & 1 \\ 1 & 0 & \cdots & & & 0 \\ 0 & 1 & 0 & & & \vdots \\ \vdots & & & & & \\ & & & \ddots & & \\ 0 & \cdots & & 0 & 1 & 0 \end{bmatrix}$$

  where the first row consists of the vector-matrices F_1, F_2; the m entries of the $1\times m$ vector F_1 are equal to zero and the rest k entries of the $1\times k$ vector F_2 are equal to one; the $(k+m-1)\times(k+m-1)$ matrix F_3 is the identity matrix and the $k+m-1$ entries of the $(k+m-1)\times 1$ vector F_4 are equal to zero.

- The $(k+m)$th degree characteristic polynomial $x(\lambda)$ *of* $F_{k,m}$ is given by

  $$x(\lambda) = \lambda^{k+m} - \sum_{i=1}^{k} \lambda^{k-i}.$$

- For the fixed integers k, m, with $k \geq 2$, $m \geq 0$, the positive numbers f_n satisfy the following limit properties:

  ✓ $\lim\limits_{n\to\infty} \dfrac{f_{n+1}}{f_n} = \rho(F_{k,m})$

  ✓ $\lim\limits_{n\to\infty} \sqrt[n]{f_n} = \rho(F_{k,m})$

where $\rho(F_{k,m})$ is the spectral radius of $F_{k,m}$ with $1 < \rho(F_{k,m}) < 2$.

References

1. M. Adam, N. Assimakis, k-step sum and m-step gap Fibonacci sequence, ISRN Discrete Mathematics, vol. 2014, Article ID 374902, 7 pages, http://dx.doi.org/10.1155/2014/374902, 2014.
2. M. Adam and N. Assimakis, k-step Fibonacci sequence and Fibonacci matrices, Journal of Discrete Mathematical Sciences & Cryptography, (October, 2015), to appear.
3. The On-Line Encyclopedia of Integer Sequences, http://oeis.org/.

Link

http://elnsite.teilam.gr/kmfib/